4

Answers and Hints

4.9 Battle of the Sexes

(a) Obvious.

(b) Let α be the probability of Alfredo going to the opera, and let β be the probability of Elisabetta going to the opera. Since in a mixed strategy equilibrium, the payoff to gambling and opera must be equal for Alfredo, we must have

$$\beta = 2(1-\beta),$$

which implies $\beta = 2/3$. Since the payoff to gambling and opera must also be equal for Elisabetta, we must have

$$2\alpha = 1-\alpha,$$

so $\alpha = 1/3$. The payoff of the game to each is then

$$\frac{2}{9}(1,2) + \frac{5}{9}(0,0) + \frac{2}{9}(2,1) = \left(\frac{2}{3}, \frac{2}{3}\right),$$

since they both go to gambling $(1/3)(2/3) = 2/9$ of the time, they both go to opera $(1/3)(2/3) = 2/9$ of the time, and otherwise they miss each other.

(c) Both do better if they can coordinate, since (2,1) and (1,2) are both better than (2/3,2/3).

Notice that we get the same answer if we find the Nash equilibrium by finding the intersection of the players' best response function. To see this, note that the payoffs to the two players are

$$\begin{aligned}\pi_1 &= \alpha\beta + 2(1-\alpha)(1-\beta) = 3\alpha\beta - 2\alpha - 2\beta + 2\\ \pi_2 &= 2\alpha\beta + (1-\alpha)(1-\beta) = 3\alpha\beta - \alpha - \beta + 1.\end{aligned}$$

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Thus

$$\frac{\partial \pi_1}{\partial \alpha} = 3\beta - 2 \begin{cases} > 0 & \text{if } \beta > 2/3 \\ = 0 & \text{if } \beta = 2/3 \\ < 0 & \text{if } \beta < 2/3 \end{cases},$$

so the optimal α is given by

$$\alpha = \begin{cases} 1 & \text{if } \beta > 2/3 \\ [0,1] & \text{if } \beta = 2/3 \\ 0 & \text{if } \beta < 2/3 \end{cases}.$$

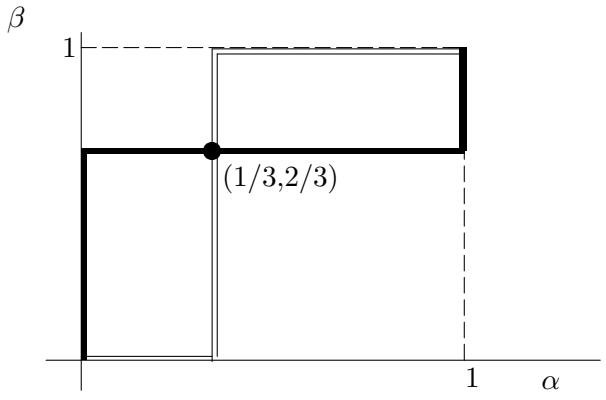
Similarly,

$$\frac{\partial \pi_2}{\partial \alpha} = 3\alpha - 1 \begin{cases} > 0 & \text{if } \alpha > 1/3 \\ = 0 & \text{if } \alpha = 1/3 \\ < 0 & \text{if } \alpha < 1/3 \end{cases},$$

so the optimal β is given by

$$\beta = \begin{cases} 1 & \text{if } \alpha > 1/3 \\ [0,1] & \text{if } \alpha = 1/3 \\ 0 & \text{if } \alpha < 1/3 \end{cases}.$$

This gives the following diagram,



4.11 One-card, Two-round Poker with Bluffing

The normal form is:

	ss	sf	f
rrbb	0 , 0	4 , -4	2 , -2
rrbf	1 , -1	0 , 0	2 , -2
rrf	2 , -2	1 , -1	0 , 0
rfbb	-5 , 5	0 , 0	2 , -2
rfbf	-4 , 4	4 , -4	2,-2
rff	-3 , 3	-3 , 3	0 , 0
fbb	-4 , 4	1 , -1	0 , 0
fbf	-3 , 3	-3 , 3	0 , 0
ff	-2 , 2	-2 , 2	-2 , 2

The last six strategies for player 1 are weakly dominated by rrbb. Eliminating these strategies gives the following reduced normal form.

	ss	sf	f
rrbb	0,0	4,-4	2,-2
rrbf	1,-1	0,0	2,-2
rrf	2,-2	1,-1	0,0

If 2 uses α ss + β sf + $(1-\alpha-\beta)f$, the payoffs to 1's strategies are:

$$\begin{aligned} &\text{rrbb:} \quad 4\beta + 2(1-\alpha-\beta) = -2\alpha + 2\beta + 2 \\ &\text{rrbf:} \quad \alpha + 2(1-\alpha-\beta) = -\alpha - 2\beta + 2 \\ &\text{rrf:} \quad 2\alpha + \beta \end{aligned}$$

If rrbb and rrbf are used, we have $\beta = \alpha/4$; if rrbb and rrf are used, we have $4\alpha = \beta + 2$. If rrbf and rrf are used we have $\alpha + \beta = 2/3$. Thus if all three are used, we have $\alpha = 8/15, \beta = 2/15$, and $1 - \alpha - \beta = 1/3$. The payoff is $18/15 = 6/5$.

If 1 uses γ rrbb + δ rrbf + $(1-\gamma-\delta)$rrf, the payoffs to 2's strategies are:

$$\begin{aligned} &\text{ss:} \quad -\delta - 2(1-\gamma-\delta) = 2\gamma + \delta - 2 \\ &\text{sf:} \quad -4\gamma - (1-\gamma-\delta) = -3\gamma + \delta - 1 \\ &\text{f:} \quad -2\gamma - 2\delta. \end{aligned}$$

Thus if ss and sf are used, $\gamma = 1/5$. If ss and f are both used, $4\gamma + 3\delta = 2$, so if all are used, $3\delta = 2 - 4/5 = 6/5$, and $\delta = 2/5$. Then $1 - \gamma - \delta = 2/5$. The payoff is 4/5 - 2 =-1/5 - 1 = -6/5, so it all works out.

There is a Nash equilibrium

$$\frac{8}{15}\text{rrbb} + \frac{2}{15}\text{rrbf} + \frac{1}{3}\text{rrf}, \qquad \frac{1}{5}\text{ss} + \frac{2}{5}\text{sf} + \frac{2}{5}\text{f},$$

with a payoff of 6/5 to player 1.

Note that we have arrived at this solution by eliminating weakly dominated strategies. Have we eliminated any Nash equilibria this way?

4.14 The Santa Fé Bar

For the first part, let p be the probability of going to the bar for each resident. The payoff to not going to the bar and the payoff to going to the bar must be equal. Thus the payoff to going to the bar must be zero. This already proves that the social surplus must be zero. To find p, note that the probability that the other two people go to the bar is p^2, so the expected payoff to going to the bar is

$$2(1-p^2)+\frac{1}{2}p^2-1=0,$$

the solution to which is $p=\sqrt{6}/3$.

For the second part, let p_i be the probability of player i going to the bar, for $i=1,\ldots,3$. In a mixed strategy equilibrium, the payoff for each player to going to the bar and staying home must be the same. It is easy to show that this is equivalent to

$$p_ip_j=\frac{a_k-c_k}{a_k-b_k}\qquad i\neq j\neq k\neq i.$$

Let $\alpha_k=p_ip_j$ for $i\neq j\neq k$, and define $\beta=\sqrt{\alpha_1\alpha_2\alpha_3}$. Notice that $\beta=p_i\alpha_i$ for $i=1,2,3$, which requires $\beta<\alpha_i$ for $i=1,\ldots,3$. This means

$$\alpha_1\alpha_2<\alpha_3$$

$$\alpha_1\alpha_3<\alpha_2$$

$$\alpha_2\alpha_3<\alpha_1.$$

This is always the case when $\alpha_1=\alpha_2=\alpha_3$.

4.16 Sex Ratios as Nash Equilibria

(a) We have

$$\alpha=\frac{d}{s}=\frac{\sigma_f(1-v)}{\sigma_m v} \tag{4.1}$$

and

$$f(u,v)=\sigma_f(1-u)c^2+\sigma_m uc^2\alpha, \tag{4.2}$$

To understand this expression, note that $\sigma_f(1-u)c$ is the number of daughters who survive to maturity, and so $\sigma_f(1-u)c^2$ is the number of grandchildren born to daughters. Similarly, $\sigma_m uc$ is the number of sons, and $\sigma_m uc(c\alpha)$ is the number of grandchildren sired by sons.

Substituting equation 4.1 into equation 4.2 and simplifying, we get

$$f(u,v) = c^2\sigma_f\left\{1 + u\left(\frac{1-2v}{v}\right)\right\}.$$

Thus if $v \neq 1/2$ there cannot be a mixed strategy equilibrium: if the fraction of males in the population is less than 50%, each female should produce all males (i.e., set $u = 1$), and if the fraction of males in the population is greater than 50%, each female should produce all females (i.e., set $u = 0$). The only possible Nash strategy is therefore $u = v = 1/2$, since such a strategy must be symmetric (the same for all agents) and mixed (since all pure strategies are clearly not Nash).

(b) Now suppose there are n females. It is easy to check that (4.1) becomes

$$\alpha = \frac{d}{s} = \frac{\sigma_f[n - u - (n-1)v]}{\sigma_m[(n-1)v + u]}.$$

The number of grandchildren (4.2) then becomes

$$\begin{aligned} f(u,v) &= \sigma_f(1-u)c^2 + \sigma_m uc^2\frac{\sigma_f[n-u-(n-1)v]}{\sigma_m[(n-1)v+u]} \\ &= \frac{c^2\sigma_f}{(n-1)v+u}\{-2u^2 - u[2(n-1)v - (n+1)] + (n-1)v\}. \end{aligned}$$

The first order condition on u for maximizing $f(u,v)$ then gives

$$2(n-1)v = n + 1 - 4u.$$

In a symmetric equilibrium, we must have $u = v$, which implies $u = v = 1/2$.

(c) Now suppose a fraction δ_m of males and δ_f of females die in each period, and the rest remain in the mating pool. Let m be the number of males and let n be the number of females in the first period. Then the ratio α of females to males in the breeding pool in the next period is given by

$$\alpha = \frac{d + n(1-\delta_f)}{s + m(1-\delta_m)} = \frac{\sigma_f cn(1-v) + n(1-\delta_f)}{\sigma_m cnv + m(1-\delta_m)}. \qquad (4.3)$$

The number of grandchildren of one female who has fraction u of males and $1-u$ of females, when the corresponding fraction for other breeding females is v, is given by

$$f(u,v) = c^2\left[\sigma_f(1-u) + \sigma_m u\alpha\right] = c^2\left\{1 + u\left[\frac{\sigma_m}{\sigma_f}\alpha - 1\right]\right\}. \quad (4.4)$$

Hence a mixed strategy Nash equilibrium requires

$$\alpha = \frac{\sigma_f}{\sigma_m}. \quad (4.5)$$

Solving (4.3) and (4.5) and simplifying, we get

$$v = \frac{1}{2}\left[1 - \frac{\sigma_f\gamma(1-\delta_m) - \sigma_m(1-\delta_f)}{\sigma_m\sigma_f c}\right], \quad (4.6)$$

where we have written $\gamma = m/n$. But in the second period m is simply the denominator of (4.3) and n is the numerator of (4.3), so (4.5) implies $\gamma = m/n = \sigma_m/\sigma_f$. Substituting this expression for γ in (4.6), we get

$$v = \frac{1}{2}\left[1 - \frac{\delta_f - \delta_m}{\sigma_f c}\right],$$

from which the result follows.

(d) Now suppose males are haploid and females are diploid. Then for female who has fraction u of sons and $1-u$ of daughters, when the corresponding fraction for other breeding females is v, the fraction of genes in daughters is $c(1-u)/2$, and the fraction in sons is cu. The number of genes (normalizing the mother's gene complement to unity) in daughters of daughters is $c^2(1-u)(1-v)/4$, the number of genes in sons of daughters is $c^2(1-u)v/2$, and the number of genes in daughters of sons is $c^2u\alpha(1-v)$. None of the female's genes are in sons of sons, since only the mother passes genetic material to her sons. The number of genes in the mother's grandchildren is the sum of these three components, which simplifies to

$$f(u,v) = c^2\left\{\frac{1+v}{4} - u\left[\frac{1+v}{4} - (1-v)\alpha\right]\right\}, \quad (4.7)$$

so we must have

$$\alpha = \frac{1+v}{4(1-v)}. \quad (4.8)$$

But by our assumption that individuals live for only one breeding period, (4.1) still holds. Solving (4.1) and (4.8) simultaneously, and defining $\nu = \sigma_f/\sigma_m$, we get

$$v = \frac{8\nu \pm \sqrt{64\nu^2 + 5}}{2(4\nu - 1)},$$

where the sign of the square root is chosen to ensure $0 < v < 1$. This implies that, for instance, that if $\sigma_f = \sigma_m$, then $v \approx 0.096$; i.e., the ratio of daughters to sons should be about ten to one.

4.18 A Mating Game

A mixed strategy for a female is a pair of probabilities (α_H, α_E), where α_H is the probability of being forward when she is H, and α_E is the probability of being forward when she is E. A mixed strategy for a male is a pair of probabilities (β_H, β_E), where β_H is the probability of being forward when he is H, and β_E is the probability of being forward when he is E. Let $\alpha = \alpha_H + \alpha_E$, and $\beta = \beta_H + \beta_E$. You can check that the payoffs for males are (a) $\pi^m_{FF} = 1$; (b) $\pi^m_{FR} = 3(2 - \alpha)/4$; (c) $\pi^m_{RF} = 3\alpha/4$; (d) $\pi^m_{RR} = 1$. The payoffs for females are (a) $\pi^f_{FF} = 1 - \beta/4$; (b) $\pi^f_{FR} = 2 - \beta$; (c) $\pi^f_{RF} = \beta/2$; (d) $\pi^f_{RR} = 1 - \beta/4$. Also, $\alpha, \beta = 2$ for FF, $\alpha, \beta = 1$ for FR and RF, and $\alpha, \beta = 0$ for RR. Now you can form the 4×4 normal form matrix, and the rest is straightforward.

4.20 Coordination Failure I

We'll do the second part. Suppose 1 uses the three pure strategies with probabilities α,β, and $\gamma = 1 - \alpha - \beta$, respectively, and 2 uses's the pure strategies with probabilities a, b, and $c = 1 - a - b$. We can assume without loss of generality that $\alpha > \beta \geq \gamma$. The payoffs to a, b and c are

$$\begin{aligned} \pi_a &= 50\beta + 40(1 - \alpha - \beta) = 40 - 40\alpha + 10\beta, \\ \pi_b &= 40\alpha + 50(1 - \alpha - \beta) = 50 + 10\alpha - 50\beta, \\ \pi_c &= 50\alpha + 40\beta. \end{aligned}$$

We have $\alpha + 2\beta > 1$, so $\beta > (1 - \alpha)/2$. Then

$$\begin{aligned} \pi_c - \pi_a &= 50\alpha + 40\beta - [40 - 40\alpha + 10\beta] = 90\alpha + 30\beta - 40 \\ &> 90\alpha + 30(1 - \alpha)/2 - 40 = 15 + 75\alpha - 40 = 75\alpha - 25 > 0. \end{aligned}$$

Thus c is better than a. Also

$$\begin{aligned}\pi_c - \pi_\beta &= 50\alpha + 40\beta - [50 - 10\alpha - 50\beta] = 60\alpha + 90\beta - 50 \\ &> 60\alpha + 90(1-\alpha)/2 - 50 = 45 + 15\alpha - 50 = 15\alpha - 5 > 0,\end{aligned}$$

so c is better than b. Thus player 2 will use c, and his payoff is $50\alpha + 40\beta > 50\alpha + 20(1-\alpha) = 20 + 30\alpha > 30$. The payoff to 1 is then $40\alpha + 50\beta > 40\alpha + 25(1-\alpha) = 25 + 15\alpha > 30$. Thus both are better off than with the 30 payoff of the Nash equilibrium.

4.21 Advertising Game

(a) It is clear that the strategies described are Nash. We will show that these are the *only* Nash equilibria in which at least one firm uses a pure strategy. Suppose first that firm 1 chooses the pure strategy M (morning). If both firms 2 and 3 choose mixed strategies, then one of them could gain by shifting to pure strategy E (evening). To see this, let the two mixed strategies be $\alpha M + (1-\alpha)E$ for firm 2 and $\beta M + (1-\beta)E$ for firm 3. Let $\pi_i(s_1 s_2 s_3)$ be the payoff to player i when the three firms use pure strategies $s_1 s_2 s_3$. Then the payoff to M for firm 2 is

$$\begin{aligned}\pi_2 &= \alpha\beta\pi_2(MMM) + \alpha(1-\beta)\pi_2(MME) + (1-\alpha)\beta\pi_2(MEM) \\ &\qquad + (1-\alpha)(1-\beta)\pi_2(MEE) \\ &= \alpha\beta(0) + \alpha(1-\beta)(0) + (1-\alpha)\beta(2) + (1-\alpha)(1-\beta)(0) \\ &= 2(1-\alpha)\beta.\end{aligned}$$

Since $0 < \beta$ by definition, this is maximized by choosing $\alpha = 0$, so firm 2 should use pure strategy E. This contradicts our assumption that both firms 1 and 2 use mixed strategies.

A similar argument holds if firm 1 uses pure strategy E. We conclude that if firm 1 uses a pure strategy, at least one of the other two firms will use a pure strategy. The firm that does will not use the same pure strategy as firm 1, since this would not be a best response. Therefore two firms use opposite pure strategies, and it doesn't matter what the third firm does.

Now we repeat the whole analysis assuming firm 2 uses a pure strategy, with clearly the same outcome. Then we do it again for firm 3.

This proves that if one firm uses a pure strategy, at least two firms use a pure strategy, which concludes this part of the problem.

(b) Let x, y, and z be the probabilities of advertising in the morning for firms 1,2, and 3. The expected return to 1 of advertising in the morning is $(1-y)(1-z)$, and in the evening it is $2yz$. If these are equal, any choice of x for firm 1 is Nash. But equality means $1-x-y-xy=0$, or $y=(1-x)/(1+x)$. Now repeat for firms 2 and 3, giving the equalities $y=(1-z)/(1+z)$ and $z=(1-x)/(1+x)$. Solving simultaneously, we get $x=y=z=\sqrt{2}-1$. To see this, substitute $y=(1-z)/(1+z)$ in $x=(1-y)/(1+y)$, getting

$$x=\frac{1-y}{1+y}=\frac{1-\frac{1-z}{1+z}}{1+\frac{1-z}{1+z}}=z.$$

Thus $x=(1-x)/(1+x)$, which is a simple quadratic equation, the only root of which between 0 and 1 is $\sqrt{2}-1$. Thus this is Nash.

To show that there are no other Nash equilibria, suppose $0<x<1$ and $0<y<1$. We must show $0<z<1$, which reproduces equilibrium (b). But $0<x<1$ implies $(1+y)(1+z)=2$ (*why?*), and $0<y<1$ implies $(1+x)(1+z)=2$. If $z=0$, then $x=y=1$, which we assumed is not the case. If $z=1$ then $x=y=0$, which is also not the case. This proves it.

4.22 Preservation of Ecology Game

The first two parts are easy. For the third and fourth, suppose x, y and z are the probabilities the three firms purify. If firm 3 purifies, its expected payoff is $-xy-x(1-y)-y(1-x)-4(1-x)(1-y)$. If firm 3 pollutes, its payoff is $-3x(1-y)-3(1-x)y-3(1-x)(1-y)$. If firm 3 is to use a mixed strategy, these must be equal, so after simplification we have $(1-3x)(1-3y)=3xy$. Solving, and repeating for the other two firms, we get the two desired solutions.

4.23 Colonel Blotto Game

The payoff matrix, giving Colonel Blotto's return (the enemy's payoff is the negative of this) is:

		Enemy Strategies (3,0)	(0,3)	(2,1)	(1,2)
	(4,0)	4	0	2	1
Colonel	(0,4)	0	4	1	2
Blotto	(3,1)	1	-1	3	0
Strategies	(1,3)	-1	1	0	3
	(2,2)	-2	-2	2	2

Suppose the enemy uses all strategies. By symmetry, 1 and 2 must be used equally, and 3 and 4 must be used equally. Let p be the probability of using (3,0), and q be the probability of using (2,1). The expected return to Colonel Blotto is then

$$\begin{aligned} 4p + \quad 2q + q &= 4p + 3q \\ 4p + q + 2q &= 4p + 3q \\ p - p + 3q &= 3q \\ -p + p + 3q &= 3q \\ -2p - 2p + 2q + 2q &= -4p + 4q. \end{aligned}$$

Colonel Blotto cannot use all strategies in a mixed strategy, since there is no p that makes all entries in this vector equal. Suppose we drop Colonel Blotto's (3,1) and (1,3) strategies, and choose p to solve $4p + 3q = -4p + 4q$ and $2p + 2q = 1$. Thus $p = 1/9$ and $q = 4/9$. Check that other solutions do not exist.

4.24 Number Guessing Game

Clearly the game is determined in the first two rounds. Let us write my strategies as (g,h,l), for "guess g, if high guess h and if low guess l. Then we have the payoff matrix (where 0 means 'not applicable')

		My Choices (102)	(103)	(213)	(310)	(320)
	1	1	1	2	2	3
Your Choices	2	2	3	1	3	2
	3	3	2	2	1	1

Suppose you choose 1,2 and 3 with probabilities α, β, and 1-$\alpha - \beta$. Then I face [-$2\alpha - \beta + 3, -\alpha + \beta + 2, 2 - \beta, \alpha + 2\beta + 1, 2\alpha + \beta + 1$]. Clearly I can't use all actions. Suppose I drop (102) and (320). Then equating

the costs of the other three, I get $\alpha = 2/5$ and $\beta = 1/5$, with cost 9/5. The costs of (102) and (320) are $2/5 + 2/5 + 6/5 = 2 > 9/5$. Therefore my choices can be part of a Nash equilibrium. Suppose I choose (103), (213), and (310) with probabilities p, q and $1-p-q$. If you use all three numbers, your payoffs are $[-p+2, -2q+3, p+q+1]$. These are equal when $p = 1/5$ and $q = 3/5$, with payoff 9/5.

4.25 Target Selection

(a) Suppose attacker uses mixed strategy $x = (x_1, \ldots, x_n)$ and defender uses strategy $y = (y_1, \ldots, y_n)$, and these form a Nash equilibrium. If $x_j = 0$, then the best response of defender must set $y_j = 0$. Suppose $x_i > 0$ for some $i > j$. Then by switching x_i and x_j, attacker gains $a_j - pa_iy_i \geq a_j - a_i > 0$.

(b) All payoffs to pure strategies used with positive probability in a best response must be equal when played against the mixed strategies of the other player(s);

(c) All payoffs to pure strategies used with positive probability in a best response must be equal when played against the mixed strategies of the other player(s);

(d) Simple algebra.

4.26 A Reconnaissance Game

The normal form matrix is:

	counter full defend	counter half defend	no counter full defend	no counter half defend
reconnoiter, full attack	$a_{11} - c + d$	$a_{12} - c + d$	$a_{11} - c$	$a_{22} - c$
reconnoiter, half attack	$a_{21} - c + d$	$a_{22} - c + d$	$a_{11} - c$	$a_{22} - c$
no reconnoiter full attack	$a_{11} + d$	$a_{12} + d$	a_{11}	a_{12}
no reconnoiter, half attack	$a_{21} + d$	$a_{22} + d$	a_{21}	a_{22}

With the given payoffs and costs, the entries in the normal form game become

46,-46	22,-22	39,-39	27,-27
10,-10	34,-34	39,-39	27,-27
55,-55	31,-31	48,-48	24,-24
19,-19	43,-43	12,-12	36,-36.

Suppose defender doesn't counter, and full defends with probability p. Then attacker faces

$$39p + 27(1-p) = 12p + 27$$

$$39p + 27(1-p) = 12p + 27$$

$$48p + 24(1-p) = 24p + 24$$

$$12p + 36(1-p) = -24p + 36.$$

Check the third and fourth. We have $-24p+36 = 24p+24$, so $p = 1/4$. Suppose attacker doesn't reconnoiter and full attacks with probability q. Then $-48q - 12(1-q) = -24q - 36(1-q)$, so $q = 1/2$. You must check that no other strategy has higher payoff, and you will find this to be true. The payoffs are $(30, -30)$. There are no other Nash equilibria, although this is tedious to prove without the appropriate computer software—which I have written just for such occasions!

4.27 Attack on Hidden Object

We have

	P	F
pp	$2\gamma - \gamma^2$	$\beta\gamma$
pF	γ	β
Fp	$\beta^2 - \beta\gamma + \gamma$	β
FF	β^2	$2\beta - \beta^2$

Note that the second row is weakly dominated by the third. Let p be the probability of defender using P. Then attacker faces

$$p(2\gamma - \gamma^2) + (1-p)\beta\gamma = p\gamma(2 - \gamma - \beta) + \beta\gamma$$

$$p(\beta^2 - \beta\gamma + \gamma) + (1-p)\beta = p(\beta^2 - \gamma\beta - \beta + \gamma) + \beta$$

$$p\beta^2 + (1-p)(2\beta - \beta^2) = -2p\beta(1-\beta) + \beta(2-\beta).$$

If attacker uses PP and PF, then

$$p = \frac{\beta(1-\gamma)}{\gamma(1-\gamma)+\beta(1-\beta)}.$$

But if attacker uses PP with probability q and PF with probability $(1-q)$, then defender faces

$$[q(2\gamma-\gamma^2)+(1-q)\gamma q\beta\gamma+(1-q)\beta] = [q\gamma(1-\gamma)+\gamma-q\beta(1-\gamma)+\beta],$$

so

$$q = \frac{\beta-\gamma}{(\gamma+\beta)(1-\gamma)}.$$

Try to find some other equilibria.

4.33 Robin Hood and Little John

(a) The payoff matrix is

	G	W
G	$-\delta-\tau_{lj}/2$ $-\delta-\tau_r/2$	0 $-\tau_r$
W	$-\tau_{lj}$ 0	$-\epsilon-\tau_{lj}/2$ $-\epsilon-\tau_r/2$

(b) The pure Nash equilibria are:

GG: $\tau_r, \tau_{lj} \geq 2\delta$;
WG: $2\delta \geq \tau_{lj}$;
GW: $2\delta \geq \tau_r$;

There is also a mixed strategy with α_r and α_{lj} the probabilities of Robin Hood and Little John going:

$$\alpha_{lj} = \frac{\epsilon+\tau_{lj}/2}{\epsilon+\delta}, \qquad \alpha_r = \frac{\epsilon+\tau_r/2}{\epsilon+\delta}$$

for $2\delta > \tau_r, \tau_{lj}$.

(c)

(d) Suppose $\tau_r > \tau_{lj}$. Then the socially optimal δ is any δ satisfying $\tau_r > 2\delta > \tau_{lj}$, since in this case it never pays to fight. The cost of crossing the bridge is τ_{lj} (or $\tau_r + 2\tau_{lj}$), including the crossing-time itself). Of course this makes Robin Hood wait all the time. He might prefer to lower or raise the costs of fighting. Will he? The payoff to the game to the players when $\tau_r > 2\delta > \tau_{lj}$ is $(-\tau_{lj}, 0)$.

Suppose Robin Hood can shift to lower-cost confrontation: we lower δ so $\tau_r > \tau_{lj} > 2\delta$. Then GG is dominant, and the gain to the two players is $(-\delta - \tau_{lj}/2, -\delta - \tau_r/2)$, which is better for Robin Hood if and only if $-\tau_{lj} < -\delta - \tau_{lj}/2$, or $2\delta < \tau_{lj}$, *which is true*!. Therefore *Robin Hood gains if he can shift to a lower-cost form of fighting.*

Suppose Robin Hood can shift to a higher cost warfare. We raise δ so $2\delta > \tau_r > \tau_{lj}$. Now the mixed strategy solution obtains, and the payoff to Robin Hood is $(-\delta - \tau_{lj}/2)(\in +\tau_{lj}/2)/(\in +\delta)$, which is easy to see is always less than $-\tau_{lj}$. Thus *Robin Hood never wants to shift to a higher-cost form of fighting*, even though he would win some of the time.

4.34 The Motorist's Dilemma

Write $\sigma = \tau/2\delta < 1$, and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ represent George and Martha's mixed strategies, where (u_1, u_2) means play G with probability u_1, play W with probability u_2, and play C with probability $1 - u_1 - u_2$. Similarly for (v_1, v_2). Let $\delta = \{(x, y)|0 \leq x, y, x + y \leq 1\}$, so δ is the strategy space for both players.[1] It is easy to check that the payoff to the pair of mixed strategies (u, v) for George is

$$f_1(u, v) = -(2\delta v_1 + (\delta + \tau/2)(v_2 - 1))u_1 - ((\delta - \tau/2)(v_1 - 1) + (\delta + \epsilon)v_2)u_2 + (\delta - \tau/2)v_1 + (\delta + \tau/2)(v_2 - 1), \quad (4.9)$$

and the payoff $f_2(u, v)$ to Martha is, by symmetry, $f_2(u, v) = f_1(v, u)$. The players reaction sets are given by

$$R_1 = \{(u, v) \in \delta \times \delta | f_1(u, v) = \max_{\mu} f_1(\mu, v)\}$$

$$R_2 = \{(u, v) \in \delta \times \delta | f_2(u, v) = \max_{\mu} f_2(\mu, v)\}.$$

and the set of Nash equilibria is $R_1 \cap R_2$.

If the coefficients of u_1 and u_2 are negative in equation 4.9, then (0,0) is the only Nash equilibrium for George.

[1] As a notational convention, we write $\{x|p(x)\}$ to mean "the set of all x such that the assertion $p(x)$ is true."

4.36 Frankie and Johnny

Let π be the payoff to Johnny, and write $\overline{x} = (x_f + x_j)/2$. If $x_f < x_j$, then $y < \overline{x}$ implies $\pi = x_f$, and otherwise $\pi = x_j$. If $x_f > x_j$, then $y < \overline{x}$ implies $\pi = x_j$, and otherwise $\pi = x_f$. Since

$$\Pr\{y < \overline{x}\} = F(\overline{x}),$$

we have

$$\pi = \begin{cases} x_f F(\overline{x}) + x_j(1 - F(\overline{x})) & \text{for } x_f < x_j; \\ x_j F(\overline{x}) + x_f(1 - F(\overline{x})) & \text{for } x_f > x_j. \end{cases}$$

If $x_f > x_j$, the first order conditions on x_f and x_j are

$$F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0,$$

$$1 - F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0,$$

from which it follows that $F(\overline{x}) = 1/2$. If $x_f < x_j$, we arrive at the same conclusion. Thus any pair (x_f, x_j) such that $F(\overline{x}) = 1/2$ is a Nash equilibrium, and the negotiator chooses randomly between the two offers.

4.37 A Card Game

The only undominated strategy for each player is to choose a critical level x_i^* and to fold if $x_i < x_i^*$. Let (x_1^*, x_2^*) be Nash strategies. The payoff to player 1 is

$$\begin{aligned} -1 \cdot \mathrm{P}[x_1 < x_1^*] + 1 \cdot \mathrm{P}[x_1 > x_1^*, x_2 < x_2^*] \\ - 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] \\ + 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1]. \end{aligned}$$

Clearly we have

$$\mathrm{P}[x_1 < x_1^*] = x_1^*, \qquad \mathrm{P}[x_1 > x_1^*, x_2 < x_2^*] = (1 - x_1^*)x_2^*.$$

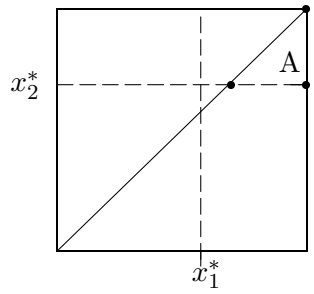
We also know

$$\begin{aligned} \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] + \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] \\ = \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*] \\ = (1 - x_1^*)(1 - x_2^*). \end{aligned}$$

To evaluate $\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, suppose $x_1^* > x_2^*$. Then

$$\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] = \mathrm{P}[x_1 > x_1^*, x_2 > x_1] = \frac{(1-x_1^*)^2}{2}$$

To see this, consider the following diagram:

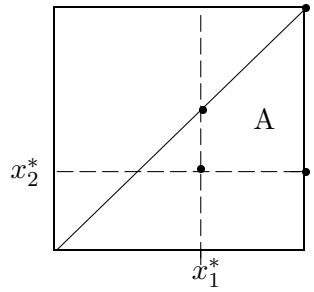


Since x_1 and x_2 are independently distributed, the pair (x_1, x_2) is uniformly distributed in the unit square depicted above. the case $\mathrm{P}[x_1 > x_1^*, x_2 > x_1]$ is the little triangle labeled 'A', which has area $(1-x_1^*)^2/2$.

We thus have

$$\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] = (1-x_1^*)(1-x_2^*) - \frac{(1-x_1^*)^2}{2}.$$

To evaluate $\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$ when $x_1^* < x_2^*$, refer to the following diagram:



Calculating the area of the trapezoid 'A' representing the case $\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, we get

$$\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_1 < x_2] = (1-x_1^*)(1-x_2^*) - \frac{(1-x_2^*)^2}{2}.$$

Suppose $x_1^* > x_2^*$. The payoff to player 1 is then

$$\begin{aligned}\pi &= -x_1^* + (1-x_1^*)x_2^* - 6\frac{(1-x_1^*)^2}{2} + 6\left[(1-x_1^*)(1-x_2^*) - \frac{(1-x_1^*)^2}{2}\right] \\ &= 5x_1^* - 5x_2^* - xx_1^{*2} + 5x_1^*x_2^*.\end{aligned}$$

The first order condition on x_2^* is then $-5 + 5x_1^* = 0$, so $x_1^* = 1$. The first order condition on x_1^* is $5 - 12x_1^* + 5x_2^* = 0$, so $x_2^* = 7/5$, which is impossible.

Thus we must have $x_1^* < x_2^*$. The payoff to player 1 is then

$$-x_1^* + (1 - x_1^*)x_2^* - 6\left[(1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_2^*)^2}{2}\right] + 6\frac{(1 - x_2^*)^2}{2},$$

which reduces to

$$5x_1^* - 5x_2^* - 7x_1^*x_2^* + 6x_2^{*2}.$$

The first order condition on x_1^* give $x_2^* = 5/7$, and the first order condition on x_2^* then gives $x_1^* = 25/49$. Note that we indeed have $x_1^* < x_2^*$. The payoff of the game to player 1 is then

$$5\frac{25}{49} - 5\frac{5}{7} + 6\left(\frac{5}{7}\right)^2 - 7\frac{25}{49}\frac{5}{7} = -25/49.$$

4.38 Cheater-Inspector

Let α be the probability of trusting. If there is a mixed strategy equilibrium in the n-round game, the payoff to cheating in the first period is $\alpha n + (1 - \alpha)(-an) = \alpha n(1 + a) - an$, and the payoff to being honest is $g_{n-1} + b(1 - \alpha)$. Equating these, we find

$$\alpha = \frac{g_{n-1} + b + an}{n(1 + a) + b},$$

assuming $g_{n-1} < n$ (which is true for $n = 0$, and which we will show is true for larger n by induction). The payoff of the n-round game is then

$$g_n = g_{n-1} + b\frac{n - gn - 1}{n(1 + a) + b}.$$

It is easy to check that $g_1 = b/(1 + a + b)$ and $g_2 = 2b/(1 + a + b)$, which suggests that

$$g_n = \frac{nb}{1 + a + b}.$$

This can be checked directly by assuming it true for g_{n-1} and proving it true for g_n (this is called Proof by Induction):

$$g_n = g_{n-1} + b\frac{n - gn - 1}{n(1 + a) + b}$$

$$
\begin{aligned}
&= \frac{b(n-1)}{1+a+b} + b\frac{n - \frac{b(n-1)}{1+a+b}}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b}\frac{n+na+nb-b(n-1)}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b} \\
&= \frac{bn}{1+a+b}.
\end{aligned}
$$

4.39 The Groucho Marx Game

(a) Stay if you get the 3 is strictly dominated by raise, so there are four strategies left: rr,rs,sr,and ss (refer to the hint in the question for the meaning of these strategies). Here are the payoffs, where down the first column, 12 means (player 1 draws 1, player 2 draws 2), etc.:

	rr/rr	rr/rs	rr/sr	rr/ss	rs/rs	rs/sr	rs/ss	sr/sr	sr/ss	ss/ss
12	-a-b	a	-a-b	a	a	-a-b	a	-a	0	0
13	-a-b	-a-b	-a-b	-a-b	-a-b	-a-b	-a-b	-a	-a	-a
21	a+b	a+b	a	a	-a	0	0	a	a	0
23	-a-b	-a-b	-a-b	-a-b	-a	-a	-a	-a-b	-a-b	-a
31	a+b	a+b	a	a	a+b	a	a	a	a	a
32	a+b	a	a+b	a	a	a+b	a	a+b	a	a
	0	2a	-2b	2(a-b)	0	-a-b	a-b	0	a-b	0

The answer follows directly from the resulting payoff matrix:

	rr	rs	sr	ss
rr	0	$a/3$	$-b/3$	$(a-b)/3$
rs	$-a/3$	0	$-(a+b)/3$	$(a-b)/3$
sr	$b/3$	$(a+b)/3$	0	$(a-b)/3$
ss	$-(a-b)/3$	$-(a-b)/3$	$-(a-b)/3$	0

(b) Staying when you pick the 4 is strictly dominated by raising. This leaves us with eight strategies for each player. Notice that staying with 2 and raising with 1 is weakly dominated by staying with 1 or 2

(explain). This generalizes to the conclusion that we only eliminate dominated strategies by staying unless the card you pick is greater than some number between zero and three. Thus four strategies remain: {rrr, srr, ssr, sss}. The payoff of any strategy against itself is clearly zero. Thus it remains to calculate:

	rrr/srr	rrr/ssr	rrr/sss	srr/ssr	srr/sss	ssr/sss
12	-a-b	a	a	0	0	0
13	-a-b	-a-b	a	-a	0	0
14	-a-b	-a-b	-a-b	-a	-a	-a
21	a	a	a	a	a	0
23	-a-b	-a-b	a	-a-b	a	0
24	-a-b	-a-b	-a-b	-a-b	-a-b	-a
31	a	a	a	a	a	a
32	a+b	a	a	a	a	a
34	-a-b	-a-b	-a-b	-a-b	-a-b	-a-b
41	a	a	a	a	a	a
42	a+b	a	a	a	a	a
43	a+b	a+b	a	a+b	a	a+b
	-3b	2a-4b	6a-3b	a-2b	4a-2b	2a

Here is 12 times the payoff matrix (only player 1's payoffs listed), from which the result follows:

	rrr	srr	ssr	sss
rrr	0	$-3b$	$2a-4b$	$6a-3b$
srr	$3b$	0	$a-2b$	$4a-2b$
ssr	$-2a+4b$	$-a+2b$	0	$2a$
sss	$6a-3b$	$-4a+2b$	$-2a$	0

(c) We represent the strategy of raising if and only if the card chosen is greater than k by s_k. Thus each player has n pure strategies (eliminating weakly dominated strategies). We must find the payoff to each pure strategy pair $\{(s_k, s_l)|k, l = 1, \ldots, n\}$. Suppose the pure strategies used are (s_k, s_l) and the cards picked from the hat by players one and two are $\tilde{k}$ and $\tilde{l}$, respectively.

First suppose $k \geq l$. The probability player one wins if both stay is

$$\mathrm{P}[\tilde{k} > \tilde{l}|\tilde{k} \leq k, \tilde{l} \leq l] = \mathrm{P}[\tilde{k} \leq l]\mathrm{P}[\tilde{k} > \tilde{l}|\tilde{k}, \tilde{l} \leq l] + \mathrm{P}[\tilde{k} > l|\tilde{k} \leq k, \tilde{l} \leq l]$$

$$= \frac{l}{k}\frac{1}{2} + \frac{k-l}{k} = 1 - \frac{l}{2k}.$$

Since the probability player one loses if both stay is one minus the above quantity, and player 1 stands to win or lose a in this case, we find that player one's expected payoff in this case is

$$\pi_{k \geq l}[\tilde{k} > \tilde{l} | \tilde{k} \leq k, \tilde{l} \leq l] = a\left(1 - \frac{l}{k}\right).$$

By symmetry (interchange k and l and then negate—or you can calculate it out), we have

$$\pi_{k < l}[\tilde{k} > \tilde{l} | \tilde{k} \leq k, \tilde{l} \leq l] = -a\left(1 - \frac{k}{l}\right).$$

We also have the following easy payoffs:

$$\pi[\tilde{k} > k, \tilde{l} \leq l] = a$$
$$\pi[\tilde{k} \leq k, \tilde{l} > l] = -a$$

Finally, suppose both players raise. First assume $k \geq l$. Then

$$\begin{aligned} \mathrm{P}[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l] &= \mathrm{P}[\tilde{k} > \tilde{l} | \tilde{k}, \tilde{l} > k] + \mathrm{P}[\tilde{l} \leq k | \tilde{l} > l] \\ &= \frac{n-k}{n-l}\frac{1}{2} + \frac{k-l}{n-l}. \end{aligned}$$

Since the probability player one loses if both raise is one minus the above quantity, and player 1 stands to win or lose $a+b$ in this case, we find that player one's expected payoff in this case is

$$\pi_{k \geq l}[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l] = (a+b)\frac{k-l}{n-l}.$$

By symmetry (or you can calculate it out), we have

$$\pi_{k < l}[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l] = (a+b)\frac{k-l}{n-k}.$$

Now we add everything up:

$$\begin{aligned} \pi_{k \geq l} = &\mathrm{P}[\tilde{k} \leq k]\mathrm{P}[\tilde{l} \leq l]\pi_{k \geq l}[\tilde{k} > \tilde{l} | \tilde{k} \leq k, \tilde{l} \leq l] \\ &+\mathrm{P}[\tilde{k} \leq k]\mathrm{P}[\tilde{l} > l]\pi[\tilde{k} \leq k, \tilde{l} > l] \\ &+\mathrm{P}[\tilde{k} > k]\mathrm{P}[\tilde{l} \leq l]\pi[\tilde{k} > k, \tilde{l} \leq l] \\ &+\mathrm{P}[\tilde{k} > k]\mathrm{P}[\tilde{l} > l]\pi_{k \geq l}[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l] \\ = &\frac{1}{n^2}(l-k)(a(k-l) - b(n-k)). \end{aligned}$$

By symmetry (or calculation if you don't trust your answer—I did it by calculation and checked it by symmetry), we have

$$\pi_{k<l} = \frac{1}{n^2}(k-l)(a(k-l)+b(n-l)).$$

4.40 Real Men Don't Eat Quiche

Let B (drink beer) and Q (eat quiche) be the actions for Clem, and B (bully Clem) and D (defer to Clem) be the actions for the Dark Stranger. The strategies for Clem are BB (drink beer), BQ (drink beer if Tough, eat Quiche if Wimp), QB (eat quiche if Tough, drink beer if Wimp), and QQ (eat quiche), and those for the Dark Stranger are BB (bully), DB (defer if Clem drinks beer, bully if Clem eats quiche), BD (bully if Clem drinks beer, defer if Clem eats quiche), and DD (defer). The payoff matrices in the Tough and Wimp cases are given in the following diagram, where $a, b/c, d$ means "Clem's payoff is a if Tough and c if Wimp; Dark Stranger's payoff is b if Clem is Tough, and d if Clem is a Wimp:

	BB	BD	DB	DD
BB	1,0/0,1	1,0/0,1	3,1/2,0	3,1/2,0
BQ	1,0/1,1	1,0/3,0	3,1/1,1	3,1/3,0
QB	0,0/0,1	2,1/0,1	0,0/2,0	2,1/2,0
QQ	0,0/1,1	2,1/3,0	0,0/1,1	2,1/3,0

The expected return to the players is thus 1/3 times the first matrix plus 2/3 times the second matrix, since 1/3 is the probability that Clem is Tough. We get the payoff matrix (all payoffs multiplied by 3):

	BB	BD	DB	DD
BB	1,2	1,2	7,1	7,1
BQ	3,2	7,0	5,3	9,1
QB	0,2	2,3	4,0	6,1
QQ	2,2	8,1	2,2	8,1

Now QB is dominated by BQ, DD is dominated by BB, and BD is dominated by BB (since QB is out). But then QQ is dominated by BQ, so we have

	BB	DB
BB	1,2	7,1
BQ	3,2	5,3

This is sensible: Clem can bluff Tough or not, and the Dark Stranger can call the bluff or not. There is a mixed strategy equilibrium. Let α be the probability of BB. The payoff to BB and DB must then be equal for the Dark Stranger, so $2 = \alpha + 3(1-\alpha)$, so $\alpha = 1/2$. If β is the probability of the Dark Stranger using BB, then $\beta + 7(1-\beta) = 3\beta + 5(1-\beta)$, or $\beta = 1/2$.

4.44 The Equivalence of Behavioral and Mixed Strategies

(a) Suppose player i has information sets $\nu_1, \ldots, \nu_k$, so a pure strategy for i can be written as $a_1 a_2 \ldots a_k$, where $a_j \in \nu_j$ for $\jmath = 1, \ldots, k$. Then

$$\begin{aligned}\sum_{s \in S_i} \alpha_s^p &= \sum_{a_1 \in \nu_1} \cdots \sum_{a_k \in \nu_k} p_i(a_1) \cdots p_i(a_k) \\ &= \sum_{a_1 \in \nu_1} p_i(a_1) \ldots \sum_{a_k \in \nu_k} p_i(a_k) = 1.\end{aligned}$$

(b) Let σ be the mixed strategy representation of behavioral strategy p. We must show that for every player i and every information set $\nu \in \mathcal{N}_i$, if $\mathrm{P}[\nu|\sigma_i] > 0$, then for every $a \in \alpha^\nu$, we have $p(a) = \mathrm{P}[a|\sigma_i]/p[\nu|\sigma_i]$. If $N \subseteq \mathcal{N}_i$, we denote by S_i/N the set of pure strategies over the nodes $\{\nu | \nu \in \mathcal{N}_i - N\}$, where for sets A and B, $A - B = \{a \in A | a \notin B\}$; i.e., $s \in S_i/N$ is a choice of a node in each of player i's information sets not in N. Similarly, we denote by $S_i/N[a]$ the pure strategies over $\mathcal{N}_i - N$ that lead to node a for some choice of pure strategies of the other players, and we denote by $S_i/N[\nu]$ the pure strategies over $\mathcal{N}_i - N$ that lead to information set ν for choice of pure strategies of the other players. Finally, $n(\mu, a)$ is the statement "$\mu \in \mathcal{N}_i$ is not in the path from the root node r to node a". Then for $\sigma_i = \sum_{s \in S_i} \alpha_s s$, $a \in \nu \in \mathcal{N}_i$, $\mu' \in N = \{\mu \in$

$\mathcal{N}_i|n(\mu,a)\}$, we have

$$\begin{aligned}
\mathrm{P}[a|\sigma_i] &= \sum_{s\in S_i[a]} \alpha_s \\
&= \sum_{s\in S_i[a]} \prod_{\mu\in\mathcal{N}_i} p_i(s_i(\mu)) \\
&= \sum_{s\in S_i[a]} p_i(s_i(\mu')) \prod_{\substack{\mu\in\mathcal{N}_i \\ \mu\neq\mu'}} p_i(s_i(\mu)) \\
&= \sum_{s\in S_i/\{\mu'\}[a]} \prod_{\substack{\mu\in\mathcal{N}_i \\ \mu\neq\mu'}} p_i(s_i(\mu)) \\
&= \sum_{s\in S_i/N[a]} \prod_{\substack{\mu\in\mathcal{N}_i \\ \mu\notin N}} p_i(s_i(\mu)) \\
&= \prod_{\substack{\mu\in\mathcal{N}_i \\ \mu\notin N}} p_i(s_i^a(\mu)),
\end{aligned}$$

where s_i^a is the unique member of S_i/N that chooses at each information set on the path from r to a, the branch belonging to that path. But the last expression is the product of the probabilities assigned by p to i's choices along the path from r to a. A similar argument shows that $\mathrm{P}[\nu|\sigma_i]$ is the sum of the product of the probabilities assigned by p to i's choices along all paths from r to nodes in ν. But by the assumption of perfect recall, i must make the same choices to get to any node in ν, so there is only one such path. It follows that, provided the denominator is not zero, the behavioral probability assigned to node a is just

$$\frac{\mathrm{P}[a|\sigma_i]}{\mathrm{P}[\nu|\sigma_i]} = p_i(a),$$

which proves that p is a behavioral representation of σ.

(c) The payoff to player i given by behavioral strategy p is

$$\pi_i(p) = \sum_{t\in T} \mathrm{P}[r,t|p]\pi_i(t),$$

and the payoff given by mixed strategy σ is

$$\pi_i(\sigma) = \sum_{t\in T} \mathrm{P}[r,t|\sigma]\pi_i(t),$$

where $\mathrm{P}[r, a|\sigma]$ is the probability of reaching a from the root node r if all players randomize their choice of pure strategies according to the weighting in σ. To complete the proof, we must show that $\mathrm{P}[r, t|\sigma] = \mathrm{P}[r, t|p]$.[2] We will show that for every node a, $\mathrm{P}[r, a|\sigma] = \mathrm{P}[r, a|p]$. This is clearly true for $a = r$, since then both sides of the equation are unity. Suppose it is true for a node a and a' is a child node of a. If Nature moves at a and chooses a' with probability q, then clearly $\mathrm{P}[r, a'|p] = q\mathrm{P}[r, a|p]$. Let $p_N(a)$ be the product of the probabilities of all the branches associated with Nature on the path from r to a. Then if $\sigma_i = \sum_{s\in S_i} \alpha_s s$, we have

$$\begin{aligned}
\mathrm{P}[r, a'|\sigma] &= p_N(a') \prod_{j=1}^{n} \sum_{s_j \in S_j[a']} \alpha_{s_j} \\
&= p_N(a) \prod_{j=1}^{n} \sum_{s_j \in S_j[a']} \alpha_{s_j} \\
&= p_N(a) \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \sum_{s_i \in S_i[a']} \right) \sum_{s_i \in S_i[a']} \alpha_{s_i} \\
&= p_N(a) \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \sum_{s_i \in S_i[a']} \right) q \sum_{s_i \in S_i[a]} \alpha_{s_i} \\
&= \mathrm{P}[r, y|\sigma] q.
\end{aligned}$$

Note that we used perfect recall in the next-to-last step, where we used the equation

$$\sum_{s_i \in S_i[a']} \alpha_{s_i} = q \sum_{s_i \in S_i[a]} \alpha_{s_i}.$$

Under perfect recall, $S_i[a] = S_i[\nu]$, where $a \in \nu$, and with this substitution, the equation is the definition of $q = p(a')$.

(d) If p_ν is not a best response, then there is some q_ν that gives i a higher payoff. Then the mixed strategy representation of $(q_\nu, p_{-\nu})$

[2]The reader will note that to avoid a proliferation of new symbols, I have been using what is known as "functional overloading," by which is meant that two functions with the same function name but with different arguments represent two distinct functions. Thus $\mathrm{P}[a|\sigma_i]$, $\mathrm{P}[\nu|\sigma_i]$, $\mathrm{P}[r, a|\sigma]$ and $\mathrm{P}[r, a|p]$ are notationally unambiguous but have distinct meanings, so long as a is a node, ν is an information set, σ_i is a mixed strategy, and r is the root node of the game tree.

gives i a higher payoff than σ_i, which contradicts the fact that σ_i is a best response.